

# MINIMAX STATE OBSERVATION IN LINEAR ONE DIMENSIONAL 2-POINT BOUNDARY VALUE PROBLEMS

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**Abstract.** In this paper we study observation problem for linear 2-point BVP  $\mathcal{D}x(\cdot) = \mathcal{B}f(\cdot)$  assuming that information about system input  $f(\cdot)$  and random noise  $\eta$  in system state observation model  $y(\cdot) = \mathcal{H}x(\cdot) + \eta$  is incomplete ( $f(\cdot)$  and  $M\eta\eta'$  are some arbitrary elements of given sets). A criterion of guaranteed (minimax) estimation error finiteness is proposed. Representations of minimax estimations are obtained in terms of 2-point BVP solutions. It is proved that in general case we can only estimate a projection of system state onto some linear manifold  $\mathcal{F}$ . In particular,  $\mathcal{F} = \mathbb{L}_2^n$  if  $\dim \mathcal{N}(\frac{\mathcal{D}}{\mathcal{H}}) = 0$ . Also we propose a procedure which decides if given linear functional belongs to  $\mathcal{F}$ .

## Problem statement

Let  $t \mapsto x(t)$  – totally continuous vector-function from space of square summable  $n$ -vector-functions  $\mathbb{L}_2^n := \mathbb{L}_2([0, \omega], \mathbb{R}^n)$  – be a solution of BVP

$$\dot{x}(t) - A(t)x(t) = B(t)f(t), x(0) = x(\omega), \quad (1)$$

where  $t \mapsto A(t)$  ( $t \mapsto B(t)$ ) –  $n \times n$  ( $n \times r$ )-matrix-valued continuous function,  $\omega < +\infty$ ,  $f(\cdot) \in \mathbb{L}_2^r$ .

We suppose that a realization of  $m$ -vector function  $t \mapsto y(t)$  is observed at  $[0, \omega]$

$$y(t) = H(t)x(t) + \eta(t), \quad (2)$$

where  $t \mapsto x(t)$  is one of the possible solutions of (1) for some  $f(\cdot) \in \mathcal{G}$ ,  $t \mapsto H(t)$  –  $m \times n$ -matrix-valued continuous function,  $t \mapsto \eta(t)$  – realization of mean-square continuous random process with zero expectation and uncertain correlation function  $(t, s) \mapsto R_\eta(t, s) \in \mathcal{G}_2$ . Let

$$\mathcal{G} := \{f(\cdot) : \int_0^\omega (f(t), f(t))dt \leq 1\},$$

$$\mathcal{G}_2 := \{R_\eta : \int_0^\omega \text{sp} R_\eta(t, t)dt \leq 1\}$$

and consider linear functional

$$\ell(x) := \int_0^\omega (\ell(t), x(t))dt, \quad \ell(\cdot) \in \mathbb{L}_2^n,$$

defined on the (1) solutions domain. We will be looking for  $\ell(x)$  estimation in terms of

$$u(y) := \int_0^\omega (u(t), y(t))dt, \quad u(\cdot) \in U_\ell \subset \mathbb{L}_2^m$$

For each  $u(\cdot)$  we associate *guaranteed estimation error*<sup>1</sup>

$$\sigma(u) := \sup_{x(\cdot) \in \mathcal{D}(\mathcal{D}), \mathcal{D}x(\cdot) \in \mathcal{G}, R_\eta \in \mathcal{G}_2} \{M[\ell(x) - u(y)]^2\}$$

**Definition 1.** Function  $\hat{u}(\cdot) \in U_\ell$  is called *minimax mean-square estimation* if it satisfies

$$\sigma(\hat{u}) \leq \sigma(u), \quad u(\cdot) \in U_\ell \quad (3)$$

Term

$$\hat{\sigma} := \inf_{u \in U_\ell} \sigma(u)$$

is called minimax mean-square error.

**Theorema 1.** *Boundary value problem*

$$\begin{aligned} \dot{z}(t) &= -A'(t)z(t) + H'(t)H(t)p(t) - \ell(t), \\ \dot{p}(t) &= A(t)p(t) + B(t)B'(t)z(t), \\ z(0) &= z(\omega), p(0) = p(\omega) \end{aligned} \quad (4)$$

has non-empty solutions domain iff

$$Ph(\omega) \perp \mathcal{N}(W(0, \omega)),$$

where  $P := [E - (E - \Phi(\omega, 0))(E - \Phi(\omega, 0))^+]$ ,  $\Phi$  – fundamental solution of  $\dot{z}(t) = -A'(t)z(t)$ ,

$$W(0, \omega) := \int_0^\omega P\Phi(\omega, s)H'(s)H(s)\Phi'(\omega, s)Pds,$$

$h(\cdot)$  is a solution of

$$\dot{h}(t) = -A'(t)h(t) + \ell(t), h(0) = 0$$

<sup>1</sup>Linear mapping  $\mathcal{D}$  is defined by the rule  $\mathcal{D}x = \dot{x} - Ax$ ,  $x \in \mathcal{D}(\mathcal{D})$ , where  $\mathcal{D}(\mathcal{D})$  is set of totally continuous vector-functions  $t \mapsto x(t)$  satisfying  $\int_0^\omega |\dot{x}(t)|_n^2 dt < +\infty$ ,  $\int_0^\omega \dot{x}(t)dt = 0$ ,  $x \mapsto Ax$  multiplies  $x(\cdot)$  by  $t \mapsto A(t)$ .

Let's illustrate theorem 1. Set

$$A(t) \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, H(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Fundamental solution  $t \mapsto F(t)$  of (1) (and fundamental solution  $t \mapsto G(t)$  of adjoint BVP)

$$F(t) \equiv \begin{pmatrix} e^t & 0 \\ -1 + e^t & 1 \end{pmatrix}, G(t) \equiv \begin{pmatrix} e^{-t} & e^{-t} - 1 \\ 0 & 1 \end{pmatrix}$$

than  $\mathcal{N}(\mathcal{D}) = \{(0, 1)\}$  and  $\mathcal{HN}(\mathcal{D}) = \{0\}$ . Let  $\ell(\cdot) = l_1(\cdot) = \begin{bmatrix} \sin(t) \\ 1 \end{bmatrix}$ . Than

$$h(t) = \begin{bmatrix} -\frac{1}{2}e^{-t}(1-2e^t+2e^t t+e^t \cos(t)-e^t \sin(t)) \\ t \end{bmatrix}$$

and

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, W(2\pi, 0) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As far as  $W(2\pi, 0)$  is a zero matrix, than according to theorem 1  $\ell(\cdot) \in \mathcal{F}$  if and only if  $Ph(2\pi) = 0$ . But for chosen  $l_1(\cdot)$

$$h(2\pi) = \begin{bmatrix} \frac{1}{2} - \frac{e^{-2\pi}}{2} - 2\pi \\ 2\pi \end{bmatrix} \Rightarrow Ph(2\pi) = \begin{bmatrix} 0 \\ 2\pi \end{bmatrix}$$

Let  $\ell(t) := l_2(t) = (\sin(t), \cos(t))$ . Than

$$h(t) = (0, \sin(t)) \Rightarrow Ph(2\pi) = (0, 0)$$

It's easy to see that (4) solution's domain is empty for  $(0, l_1(\cdot))$ . Really, null-space of adjoint BVP is  $N = \{(0, 0, 0, 1)\}$  and  $(0, l_1(\cdot))$  is not orthogonal to  $N$  while  $(0, l_2(\cdot)) \perp N$ .

Let's denote by  $\mathcal{F}$  set of all  $\ell(\cdot) \in \mathbb{L}_2^n$  satisfying condition of the theorem 1. In the next theorem we state that minimax error is finite iff  $\ell(\cdot) \in \mathcal{F}$  and in that case unique minimax estimation  $\hat{u}(\cdot)$  exists.

**Theorema 2.** *Minimax mean-square error*

$$\hat{\sigma} = \begin{cases} +\infty, & \ell(\cdot) \notin \mathcal{F}, \\ \int_0^\omega (\ell(t), \hat{p}(t))_n dt & \end{cases}$$

If  $\ell(\cdot) \in \mathcal{F}$  than unique minimax estimation  $\hat{u}(\cdot)$  exists and

$$\hat{u}(t) = H(t)\hat{p}(t),$$

where  $\hat{p}(\cdot)$  is one of the (4) solutions.

**Corollary 1.** *For given  $y(\cdot) \in \mathbb{L}_2^m$  minimax estimation  $\hat{u}(\cdot)$  can be represented as*

$$\int_0^\omega (\hat{u}(t), y(t)) dt = \int_0^\omega (\ell(t), \hat{x}(t)) dt,$$

where  $\hat{x}(\cdot)$  is any solution of

$$\begin{aligned} \dot{p}(t) &= -A'(t)p(t) - H'(t)(y(t) - H(t)x(t)), \\ \dot{x}(t) &= A(t)x(t) + B(t)B'(t)p(t), \\ p(0) &= p(\omega), x(0) = x(\omega) \end{aligned} \quad (5)$$

**Corollary 2.** *If system of functions<sup>2</sup>  $\{\mathcal{H}\psi_k(\cdot)\}$  is linear independent, than for all  $\ell(\cdot) \in \mathbb{L}_2^n$  minimax estimation is represented in terms of theorem 2 or previous corollary.*

**Corollary 3.** *If  $L$  is linear Noether closed mapping in  $\mathbb{L}_2^n$ ,  $\mathcal{H}, \mathcal{B}$  are bounded linear mappings in  $\mathbb{L}_2^n$  than*

$$(0, \ell) \in \mathcal{R} \begin{pmatrix} -L & \mathcal{B}\mathcal{B}' \\ \mathcal{H}'\mathcal{H} & L' \end{pmatrix} \Leftrightarrow \ell(\cdot) = L'z + \mathcal{H}'u(\cdot)$$

for some  $z(\cdot), u(\cdot) \in \mathbb{L}_2^n$ .

**Example 1.** We will apply corollary 1 to linear oscillator's state estimation problem

$$A(t) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H(t) \equiv \begin{pmatrix} \frac{\cos t}{2} & \frac{\sin t}{2} \\ \frac{\cos t}{2} & \frac{\sin t}{2} \end{pmatrix}$$

It's easy to see that

$$\mathcal{N}(\mathcal{D}) = \{\{\cos(t), -\sin(t)\}, \{\sin(t), \cos(t)\}\},$$

hence

$$\mathcal{HN}(\mathcal{D}) = \{\{0, 0\}, \{\frac{1}{20}, \frac{1}{2}\}\}$$

Let  $f(t) = \begin{pmatrix} \frac{\cos(t)}{\frac{\pi}{\sin(t)}} \\ \frac{\pi}{\sin(t)} \end{pmatrix}$  and suppose

$$x(t) = \frac{\cos(t)/2 + \sin(t) + t \sin(t)/\pi}{\cos(t) + t \cos(t)/\pi - \sin(t)/2}$$

is observed while noise  $g(t) = \begin{pmatrix} 0.1 \sin(t) \\ 0.1 \sin(t) \end{pmatrix}$ .

Than output  $y(t) = ((0.05 + 0.0159155t +$

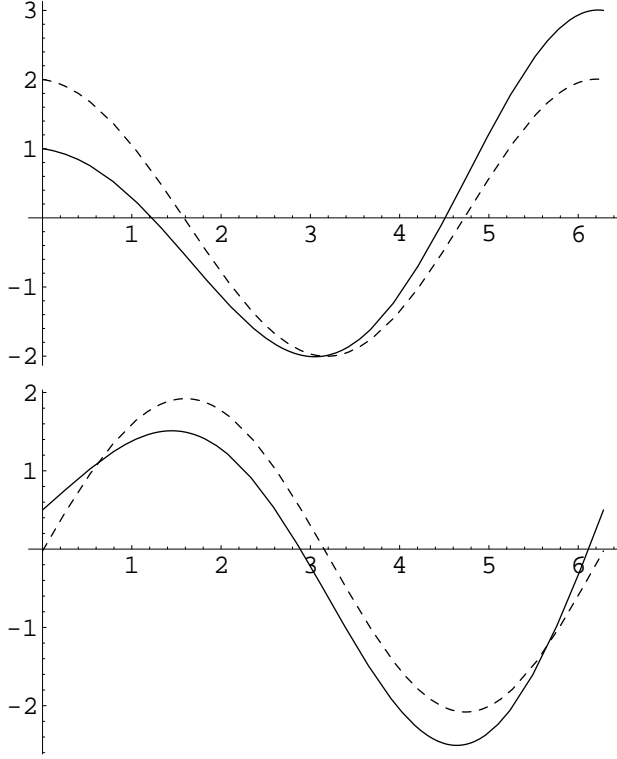
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<sup>2</sup> $\mathcal{H}\psi_k(t) = H(t)\psi_k(t)$ ,  $\psi_k(\cdot)$  are linearly independent solutions of the homogeneous BVP (1).

$0.1 \sin(t), 0.5 + 0.159155t + 0.1 \sin(t)$ , so we do not have any info about component from  $\mathcal{D}$  kernel  $(\cos(t)/2, -\sin(t)/2)$  included in  $x(t)$ . Let's find  $\hat{x}(\cdot)$  from (5). We obtain

$$\|x(\cdot) - \hat{x}(\cdot)\|_2 \simeq 1.85877$$

and  $(x(\cdot) - \text{solid line}, \hat{x}(\cdot) - \text{dashed line})$



According to theorem 2 in general case we can only estimate a projection of (1) state onto linear manifold  $\mathcal{F}$ . In particular, if  $\mathcal{N}(\mathcal{H}) \cap \mathcal{N}(\mathcal{D}) = 0$ , then  $\mathcal{F} = \mathbb{L}_2^n$  hence  $\hat{x}(\cdot)$  gives an minimax estimation of (1) state. Last condition in case of stationary matrixes  $H(t), C(t)$  means that system (1) is full observable hence this result coincides with well-known theorems of linear systems observability.